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Existentially closed models of some class of differential-difference fields

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Abstract

This paper is a study of existentially closed models of the class of differential fields with a cyclic automorphism. The main part of this paper is a report of previous studies done by Sjögren in [2], Masuoka and the author in [4]. The auther believes that the theory of iterative q-difference fields of characteristic zero has a model companion. In this paper we conjecture that, more generally, the theory of differential fields with a cyclic automorphism has a model companion. It will be explained how the conjecture relates to the study of q-difference fields.

1 Introduction

The theories of differential and difference fields have played an important role in the development of the stability theory in the model theory. In particular they give concrete examples of stability/simplicity classes. For example, the theory of differentially closed fields of characteristic zero, DCF_0 is an example of ω-stable theory and the theory of fields with a generic automorphism, ACFA, which is a theory of model companion of the theory of difference fields is an example of simple theory. The theory of differential fields of characteristic zero with an automorphism has also a model companion, $DCFA_0$ and it is also an example of simple theory. In this paper, we deal with the theory of differential fields of characteristic zero with a cyclic automorphism. The cyclic condition, in field extention, restrict extention of automorphism. For example, under anything not assume conditions, the difference field $(\mathbb{Q}, \mathrm{id}_{\mathbb{Q}})$ has two extensions to the field $\mathbb{Q}(\sqrt{2})$. On the other hand, under the cyclic condition of order 3, there is only one extension. Therefore, it is considered that the model companion of the theory of differential fields with a cyclic automorphism has a representation far from $DCFA_0$. Sjögren shown that the theory of fields of characteristic zero with a cyclic automorphism has a model companion in [2]. We approach to the theory of differential fields of characteristic zero with a cyclic automorphism by modifying the discussion of Sjögren.

This paper is organised as follows. In Section 2, we recall the definiton of differential fields, and several basic model theoretic notions as preliminaries. In Section 3, we summarize the Sjögren's results adout model companion of the theory of fields of characteristic zero with a cyclic automorphism, and cnjecture that the theory of differential felds with a cyclic automorphism has a model companion. In Section 4, we describe a related topic which a model companion of the theory of iterative q-difference fields of characteristic zero.

2 Preliminaries

2.1 Differential fields

Suppose that K is a field. An additive map $\delta : K \to K$ is a *derivation* on K if it satisfies the Leibniz rule

$$\delta(xy) = x\delta(y) + \delta(x)y,$$

We say that K is a differential field if it is equipped with a derivations. Let $(K_i; \delta_i)(i = 1, 2)$ be differential fields. We say that a field homomorphism $\sigma : K_1 \to K_2$ is *differential homomorphism* if $\sigma(\delta_1(x)) = \delta_2(\sigma(x))$ for all $x \in K_1$.

Definition 2.1. 1. The theory of differential fields with an automorphism, DF_σ, in the language $\{+, -, \times, 0, 1, \delta, \sigma\}$ consists of sentences that describe the meaning of the following

- a field,
- δ is a derivation, and
- σ is a differential automorphism.

2. Let N be a nonzero natural number. The theory of differential fields with a cyclic automorphism of order N, DF_{C_N} , is the theory $\mathrm{DF}_\sigma \cup \{\forall x(\sigma^N(x) = x)\}$.

2.2 Model companion

Let L be a first-order language and T a theory in L. We say that T is model complete if for any models M, N of T, M is an elementary submodel of N whenever M is a substructure of N. Suppose that S is an another theory in L. We say that S is a companion of T if

1. every model of T has an extension which is a model of S, and
2. every model of S has an extension which is a model of T.

We say that S is a model companion of T if

1. S is model complete, and
2. S is a companion.

If there is such a theory S, we say that T has a model companion.

Lemma 2.2. *Let T be a theory in L, M a model of T and A a subset of M. If T has a model companion then $T \cup \mathrm{Diag}(A)$ has.*

Proof. Suppose S is a model companion of T. Then, a model companion of $T \cup \mathrm{Diag}(A)$ is $S \cup \mathrm{Diag}(A)$. □

3 The theory of diferential fields with a cyclic automorphism

In this section, we describe about properties of existentially closed models of DF_{C_N} along in Sjögren's paper.

3.1 Pseudo differentially closed

Definition 3.1. Suppose that K is a differential field. We say that K is *pseudo differentially closed* if for any irreducible differential variety V over K^{alg}, V is called absolutely irreducible over K, and any differential field extension K' of K, if V has K'-rational point then V has a K-rational point.

From now, let (K, δ, σ) be an existentially closed model of DF_{C_N} and $F = \mathrm{Fix}(K, \sigma)$, fixed field of σ in K.

Theorem 3.2. *F is a peudo differentially closed field.*

Proof. Let V be an absolutely irreducible differential variety over F. Define actions of δ and σ on $K \bigotimes_F F(V)$, where $F(V)$ is the differential function field, by

$$\delta(a \otimes x) := \delta(a)b + a\delta(b), \quad \sigma(a \otimes b) := \sigma(a) \otimes b \quad (a \in K, b \in F).$$

These actions can extend uniquely to the field of fractions $K(V)$ (since V is absolutely irreducible, $K \bigotimes_F F(V)$ is an integral domain). Henece, $K(V)$ is a extention of K and a model of DF_{C_N}. Now, $x \in F[x]/I(V)$ is a $K(V)$-rational point and fixed by σ, that is,

$$K(V) \models \exists x(x \in V \wedge \sigma(x) = x).$$

Sinece K is existentially closed, we get

$$K \models \exists x(x \in V \wedge \sigma(x) = x).$$

This means that there is F-rational point, therefore F is peudo differentially closed. □

Theorem 3.3. *K is a peudo differentially closed field.*

Proof. Let V be a absoutely differential irreducible variety over K. Then, for each $i < N$, $V^{\sigma^i} = \{x : \sigma^i(f(x)) = 0, f \in I(V)\}$ is also absoutely differential irreducible variety over K. In large differentially closed field extending K, choose a_{σ^i} for every $i < N$ such that a_{σ^i} is a generic point of V^{σ^i} over $K(a_{\sigma^i} : j \neq i)$. This choice is possible because V^{σ^i} are absoutely differential irreducible varieties over K. Set $L = K(a_{\sigma^i} : i < N)$, and define action of σ by

$$\sigma(\delta^n(a_{\sigma^i})) = \delta^n(a_{\sigma^{i+1}}), \quad (i < N, m \in \mathbb{N}).$$

That makes L a model of DF_{C_N} extending K and $V(L) \neq \emptyset$. Since K is existentially closed, there is K-rational point of V , that is K is peudo differentially closed. □

3.2 Galois group

Suppose K is a differential field and F is a subfield of K. The Galois group $\mathrm{Gal}(K/F)$ of K over F is the group of all elements of automorphism of K that fixes F pointwise. The absolute Galois group $G(K)$ of K is the Galois group $\mathrm{Gal}(K^{alg}/K)$. The differential Galois group $\mathrm{Gal}_\delta(K/F)$ of K over F is the group of all elements of differential automorphism of K that fixes F pointwise. By Leibniz rule, there is the unique derivation of K^{alg} extends one of K, $G_\delta(K) := \mathrm{Gal}_\delta(K^{alg}/K)$ coinsides with $G_\delta(K)$.

Therefore, the following theorems are hold, and these proofs are the same way with Sjögren's.

Suppose that (K, δ, σ) is an existentially closed model of DF_{C_N} and $F = \mathrm{Fix}(K, \sigma)$, fixed field of σ in K.

Theorem 3.4 (ref. Theorem 4 6 in [2]). *1. $\mathrm{Gal}_\delta(K/F)$ is the cyclic group C_N of order N.*

2. The absolute Galois group of F is the universal Frattini cover of C_N.

3. The absolute Galois group of K is homeomorphic to the kernel of the universal Frattini cover of C_N.

Theorem 3.5 (ref. Theorem 10 in [2]). *Suppose that K is a model of DF_{C_N} satisfying the conditions of theorem 3.4. Then, there are no models of DF_{C_N} such that it is algebraic over K.*

3.3 Conjecture

These results, we make the following conjecture;

Conjecture it A model (K, δ, σ) is an existentially closed model of DF_{C_N} if and only if it satisfies the following conditions

1. K and $F = \mathrm{Fix}(K, \sigma)$ are pseudo differentially closed,
2. $\mathrm{Gal}_\delta(K/F) \simeq C_N$,
3. $\mathrm{Gal}_\delta(F^{alg}/F) \simeq \mathrm{Gal}_\delta(K^{alg}/K) \simeq \mathbb{Z}_N$.

4 Related topic

The notion of iterative q-difference fields was suggested by Hardouin in [3]. Iterative q-difference operator is a kind of noncommutative higher derivation. Masuoka and the author showed in [4] that there is a relationship between iterative q-difference fields and differential fields with a cyclic automorphism. In this section, we describe the relationship.

4.1 q-numbers

Let C be a field and choose an arbitrary nonzero element q in C. Let $\mathbb{F}_0$ denote the prime field included in C, and set $\mathbb{F} = \mathbb{F}_0(q)$, the subfield of C generated by q over $\mathbb{F}_0$. Following [3] we denote the q-integer, the q-factorial and q-binomial, respectively by

$$[k]_q = \frac{q^k - 1}{q - 1}, \quad [0]_q = 0,$$
$$[k]_q! = [k]_q[k-1]_q \cdots [1]_q, \quad [0]_q! = 1,$$
$$\binom{m}{n}_q = \frac{[m]_q!}{[n]_q![m-n]_q!},$$

where $k, m, n \in \mathbb{N}$ with $m > n$.

4.2 Iterative q-difference fields

Definition 4.1 (Hardouin [3]). Suppose that K is a field containing $C(t)$ and $\sigma_q : K \to K$ is a field automorphism such that it is an extension of the q-difference operator $f(t) \mapsto f(qt)$ on $C(t)$. An iterative q-difference operator on K is a sequence $\delta_K^* = (\delta_K^{(k)})_{k \in \mathbb{N}}$ of maps $\delta_K^{(k)} : K \to K$ such that

1. $\delta_K^{(0)} = \mathrm{id}_K$,
2. $\delta_K^{(1)} = \dfrac{1}{(q-1)t}(\sigma_q - \mathrm{id}_K)$,
3. $\delta_K^{(k)}(x + y) = \delta_K^{(k)}(x) + \delta_K^{(k)}(y), \ x, y \in K$,
4. $\delta_K^{(k)}(xy) = \sum_{i+j=k} \sigma_q^i(\delta_K^{(j)}(x))\delta_K^{(i)}(y), \ x, y \in K$,
5. $\delta_K^{(i)} \circ \delta_K^{(j)} = \binom{i+j}{i}_q \delta_K^{(i+j)}$.

An iterative q-difference field is field $K \supset C(t)$ given σ_q, δ_K^* such as above.

Remark 4.2. Assume that q is not a root of unity. Then, $[k]_q \neq 0$ for all $k > 0$. If $\delta_K^* = (\delta_K^{(k)})_{k \in \mathbb{N}}$ is an iterative q-difference operator on K, conditions (1), (2) and (5) above require

$$\delta_K^{(1)} = \frac{1}{(q-1)t}(\sigma_q - \mathrm{id}_K), \quad \delta_K^{(k)} = \frac{1}{[k]_q!}(\delta_k^{(1)})^k, \quad k \in \mathbb{N}.$$

Conversely, if we define $\delta_K^{(k)}$ by above, then $\delta_K^* = (\delta_K^{(k)})_{k\in\mathbb{N}}$ forms an iterative q-difference operator on K, especially, condition (4) is satisfied since one sees $\delta_K^{(1)} \circ \sigma_q = q\sigma_q \circ \delta_K^{(1)}$. Therefore, under the assumption, an iterative q-difference field is nothing but a difference field (K, σ_q).

Therefore, we assume that q is a root of unity of order N.

Lemma 4.3 ([4]). *1. For any iterative q-difference field $(K, (\delta_K^{(k)})_{k\in\mathbb{N}})$, the q-difference operator σ_q on K is of order N, that is $\sigma_K^N = \mathrm{id}_K$.*

2. There is the smallest iterative q-difference field $\mathbb{F}(t)$.

Suppose that IqDF is the theory of iterative q-difference fields.

Theorem 4.4 ([4]). *There is a functor*

$$\mathcal{F} : \{\textit{IqD-fields}\} \to \{\textit{models of } DF_\sigma\}$$

and satisfies the following properties:

1. $\mathcal{F}$ is a strictly embedding,

2. for any model (K, σ) of DF_σ there is $\mathcal{F}^{-1}(K)$ whenever $K \supset \mathcal{F}(F(t))$ and $\sigma^N = id_k$.

Moreover, by Lemma 2.2, if DF_{C_N} *has a model companion, then IqDF also admits a model companion.*

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